EXPLICIT CONSTRUCTIONS OF UNIVERSAL ℝ-TREES AND ASYMPTOTIC GEOMETRY OF HYPERBOLIC SPACES

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ABSTRACT. We present some explicit constructions of universal ℝ-trees with applications to the asymptotic geometry of hyperbolic spaces. In particular, we show that any asymptotic cone of a complete simply connected manifold of negative curvature is a complete homogeneous ℝ-tree with the valency $2^{\aleph_0}$ at every point. It implies that all these asymptotic cones are isometric depending neither on a manifold nor on an ultrafilter. It is also proved that the same tree can be isometrically embedded at infinity into such a manifold or into a non-abelian free group.

1. INTRODUCTION AND MAIN RESULTS

1.1. **Universal ℝ-trees as functional spaces.** The study of the ℝ-trees and their applications to different areas of geometry and topology has been very intensive in the last years (see a recent review [Bes] and references within). In particular, as it was pointed out by Gromov, the ℝ-trees appear naturally in the asymptotic geometry of hyperbolic metric spaces ([Gr1],[Gr2]). The aim of this paper is to present some explicit constructions related to this observation.

Recall that a metric space T is *geodesic* (see [GH]) if for every two points $t_1, t_2 \in T$ with the distance $a = d_T(t_1, t_2)$ between them, there exists an isometric inclusion $F : [0, a] \to T$, such that $F(0) = t_1$, $F(a) = t_2$. The image $F([0, a]) = [t_1, t_2]$ is called a *geodesic segment* joining the points t_1 and t_2.

A metric space T is a *real tree* (or ℝ-*tree* for short) if any pair of its points can be connected with a unique geodesic segment, and if the union of any two geodesic segments $[x, y], [y, z] \subset T$ having the only endpoint y in common is the geodesic segment $[x, z] \in T$.

The *valency* of a point $t \in T$ is the cardinal number of the set of connected components of $T \setminus t$. An ℝ-tree is μ-*universal* (see [MNO]) if any ℝ-tree with the valency not greater than μ at every its point can be isometrically embedded into it. We introduce an explicit construction of μ-universal ℝ-trees as spaces of functions.

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For every cardinal $\mu \geq 2$ let C_μ be a set such that $\mathrm{Card}(C_\mu) = \mu$ if C_μ is an infinite set and $\mathrm{Card}(C_\mu) = \mu - 1$ otherwise. Let us assume that each set C_μ contains 0 as one of it elements.

Definition 1.1.1. Let A_μ be a set of functions $f : (-\infty, \rho_f) \to C_\mu$ with the following properties: 1) for every function f there exists $\tau_f \leq \rho_f$ such that $f(t) \equiv 0 \;\forall t < \tau_f$; 2) all functions f are "piecewise–constant" from the right, i.e. for each $t \in (-\infty, \rho_f)$ there exists $\varepsilon > 0$ such that $f|_{[t,t+\varepsilon]} \equiv \mathrm{const}$. Denote the distance between two functions in A_μ by

$$d_{A_\mu}(f_1, f_2) = (\rho_{f_1} - s) + (\rho_{f_2} - s), \qquad (1.1.2)$$

where s is the *separation moment* between the functions f_1, f_2 (see [PSh]):

$$s = \sup\{t | f_1(t') = f_2(t') \quad \forall t' < t\}.$$

A relation similar to (1.1.2) in [MNO] is called a "railroad-track" equation.

For every μ the space A_μ is a complete real tree such that every its point has valency μ (see section 2.1). In particular, if $\mu = 2$, i.e. if the set $C_\mu = \{0\}$, the space A_μ is isometric to the real line. Functional spaces similar to the space A_μ were also considered in [PSh], [Ber], [Sh].

Theorem 1.1.3. (cf. [N], [MNO]). *Consider any cardinal $\mu \geq 2$.*
(i) (universality) The metric space A_μ is a μ-universal $\mathbb{R}$-tree.
(ii) (uniqueness) Any complete $\mathbb{R}$-tree with valency μ at every its point is isometric
*   to the space A_μ.*
(iii) (homogeneity) The metric space A_μ is homogeneous.

This theorem is proved in section 2.2 using simpler arguments than the analogous statements in ([N], [MNO]).
Universal $\mathbb{R}$-trees can be also constructed in a slightly different way.

Definition 1.1.4. Let μ be an infinite cardinal and let $\tilde{A}_\mu$ be a set of functions $f : [0, \rho_f) \to C_\mu$, $\rho_f \geq 0$ with the metric (1.1.2) which are piecewise-constant from the right. In particular, if $\rho_f = 0$ we consider the function $f = f_\emptyset \in \tilde{A}_\mu$. For $\mu = 2^{\aleph_0}$ we set $C_\mu = [0, 1)$ and denote $\tilde{A}_\mu$ simply by A (note that the definition of the space A given in [DP] contains an inaccuracy).

Such a representation of universal $\mathbb{R}$-trees is more convenient for applications to asymptotic geometry of hyperbolic spaces. Let us mention that for any infinite cardinal μ the spaces A_μ and $\tilde{A}_\mu$ are isometric due to part (ii) of Theorem 1.1.3 being complete $\mathbb{R}$-trees with the valency

μ at every point. Note that this would fail for μ finite: in this case the valency of the point $f_\emptyset \in \tilde{A}_\mu$ would be equal to $\mu - 1$.

1.2. **Isometric embedding at infinity.** Let X be a metric space with a distance function d_X. Intuitively, structure at infinity of the space X is what is seen if one looks at the space X from an infinitely far point (see [Gr2]). One of the possible ways to treat this notion rigorously is as follows.

Definition 1.2.1. A metric space (T, d_T) admits an *isometric embedding at infinity* into the space (X, d_X) if there exists a sequence of positive scaling factors $\lambda_i \to \infty$ such that for every point $t \in T$ one may find an infinite sequence $\{x_t^i\}$, $i = 1, 2, ..$ of points in X satisfying the relation

$$\lim_{i \to \infty} \frac{d_X(x_{t_1}^i, x_{t_2}^i)}{\lambda_i} = d_T(t_1, t_2) \qquad (1.2.2)$$

for every $t_1, t_2 \in T$.

In other words, every point of the space T corresponds to a sequence of points in X tending to infinity, such that the "normalized" pairwise distances between the sequences in X tend to the distances between the corresponding points in T. Isometric embedding at infinity is a strengthening of the notion of an *asymptotic subcone* of a metric space (see [Gr1], [GhH]).

Theorem 1.2.3. *A $2^{\aleph_0}$-universal ℝ tree can be isometrically embedded at infinity into:*
   (a) a complete simply connected manifold of negative curvature;
   (b) a non-abelian free group.

As usual, it is assumed in (a) that the curvature of a complete simply connected manifold is separated from zero. In the sequel we refer to such manifolds just as to *manifolds of negative curvature*. We prove part (a) in section 3.2.

Part (b) of this theorem is proved in section 3.1. We believe that its statement should remain true for any non-elementary hyperbolic group. Note that eventhough there exists a bilipschitz embedding of a non-abelian free group into a non-elementary hyperbolic group (see [Kap]), this is not enough to generalize (b) since isometric embeddings at infinity are not (at least, automatically) preserved by bilipschitz mappings.

Let us mention that Theorem 1.2.3 gives a complete description of metric spaces admitting an isometric embedding at infinity into negatively curved manifolds or non-abelian free groups: any such (geodesic) space is a subtree of a $2^{\aleph_0}$-universal ℝ-tree (cf. [Gr1], [GhH]).

1.3. **Asymptotic cones.** Let us recall the notion of an asymptotic cone of a metric space. A *non-principal ultrafilter* ω is a finitely additive measure on subsets of $\mathbb{N}$ such that each subset has measure either 0 or 1 and all finite subsets have measure 0. For any bounded function $h : \mathbb{N} \to \mathbb{R}$ its limit $h(\omega)$ with respect to a non-principal ultrafilter ω is uniquely defined by the following condition: for every $\varepsilon > 0$

$$\omega(\{i \in I \,|\, |h(i) - h(\omega)\| < \varepsilon\}) = 1.$$

Definition 1.3.1. (See [Gr2], [DrW], [Dru]). Fix a sequence of basepoints $O_i \in X$ and a sequence of scaling factors $\lambda_i \to \infty$, $\lambda_i \in \mathbb{R}$. Consider a set of sequences $g : \mathbb{N} \to X$ such that $d_X(O_i, g(i)) \leq \text{const}_g \cdot \lambda_i$. To any pair of such sequences g_1, g_2 one may correspond a function

$$h_{g_1,g_2}(i) = \frac{d_X(g_1(i), g_2(i))}{\lambda_i}.$$

We say that the sequences g_1, g_2 are equivalent if the limit $h_{g_1,g_2}(\omega) = 0$. The set T of all equivalence classes with the distance $d_T(g_1, g_2) = h_{g_1,g_2}(\omega)$ is an *asymptotic cone* of X with respect to the non-principal ultrafilter ω, sequence of basepoints O_i and scaling factors λ_i. We denote it by $T = \text{Con}_\omega(X, O_i, \lambda_i)$.

It was proved by M. Gromov that any asymptotic cone of a hyperbolic metric space is an $\mathbb{R}$-tree (see [Gr1],[Gr2])

Theorem 1.3.2. *Any asymptotic cone of a manifold of negative curvature is isometric to the $2^{\aleph_0}$-universal $\mathbb{R}$-tree.*

We prove this theorem in section 2.3. Theorem 1.3.2 together with Theorem 1.2.3 imply that any asymptotic cone of a manifold of negative curvature can be isometrically embedded at infinity into such manifold.

It follows from Theorem 1.3.2 that asymptotic cones of manifolds of negative curvature depend neither on a manifold nor an ultrafilter or the sequence of scaling factors. The statement of Theorem 1.3.2 is folklore for any non-elementary hyperbolic group (cf. [KapL], [Dru]). Let us note, however, that in general asymptotic cones do depend on the choice of ultrafilters: in particular, there exists a finitely generated group with non-homeomorphic asymptotic cones corresponding to different ultrafilters (see [ThV]).

Main results of the present paper were announced in [DP].

2. Universal $\mathbb{R}$-trees and asymptotic cones

2.1. **Properties of the space A_μ.**

Lemma 2.1.1. *The space A_μ is a real tree such that the valency of every its point is μ.*

Proof. It is easy to see that the space A_μ with the metric (1.1.2) is a real tree. Let us prove that the valency of each point is μ. Consider a function $f : (-\infty, \rho_f) \to C_\mu$ in A_μ. For any $c \in C_\mu$ take a set of functions $f_\delta : (-\infty, \rho_f + \delta) \to C_\mu$ such that $f_\delta(t) = f(t)$ if $-\infty < t < \rho_f$ and $f_\delta(t) \equiv c$ if $\rho_f \le t < \rho_f + \delta$. For different constants c we get non–intersecting rays in A_μ starting from the point f (note that if μ is finite the total number of rays emanating from f is exactly $\mathrm{Card}(C_\mu) + 1 = \mu$). Since A_μ is a real tree, these rays lie in different connected components of $A_\mu \setminus f$. Therefore, A_μ has valency μ at every its point. $\square$

Lemma 2.1.2. *The metric space A_μ is complete.*

Proof. Consider a fundamental sequence $f_i : (-\infty, \rho_{f_i}) \to C_\mu$ in A_μ. Then $\rho_i = \rho_{f_i}$ is also fundamental, since $d(f_i, f_j) \ge |\rho_i - \rho_j|$. Denote $\rho = \lim_{i \to \infty} \rho_i$. Then $\lim_{i,j \to \infty} s_{ij} = \rho$, where s_{ij} is a separation moment of functions f_i, f_j. Indeed, $d_{A_\mu}(f_i, f_j) = (\rho_i - s_{ij}) + (\rho_j - s_{ij})$ and hence

$$s_{ij} = \frac{\rho_i + \rho_j - d_{A_\mu}(f_i, f_j)}{2} \to \rho.$$

Therefore for each $\rho' < \rho$ the functions f_i coincide for sufficiently large i on the interval $(-\infty, \rho')$. Define f by $f(x) = \lim_{i \to \infty} f_i(x)$ for any $x \in (-\infty, \rho)$. Note that $f \in A_\mu$, since for any $x \in (-\infty, \rho)$ there exist $I \in \mathbb{N}$ and $\varepsilon > 0$, such that $f|_{(-\infty, x+\varepsilon]} \equiv f_I|_{(-\infty, x+\varepsilon]}$ and hence f is "piecewise–constant" from the right. $\square$

2.2. **Proof of Theorem 1.1.3.** (i) Let Ψ be a tree of cardinality $card(\Psi) \le \mu$. We say that a subtree $T \subset \Psi$ is *good* if it satisfies the following two properties: a) T is nonempty and connected; b)$\forall t \in T$ either T has a nonempty intersection with all connected components of the set $\Psi \setminus t$ (in this case we say that t is *full-valented*), or T has a non–empty intersection with at most 2 connected components of $\Psi \setminus t$ (then we call t *few–valented*). Note that union of the nested good subtrees is also a good subtree. Consider pairs $P = (T, F)$ such that T is a good subtree and $F : T \to A_\mu$ is an isometric inclusion. Let us introduce a partial order on such pairs: $(T_1, F_1) \le (T_2, F_2)$ if $T_1 \subset T_2$ and $F_2|_{T_1} = F_1$. The set of such pairs is nonempty since it contains all inclusions of a one-point set. Any linearly ordered set of pairs $P_\alpha = (T_\alpha, F_\alpha)$ has a maximal element (T', F'), where $T' = \cup T_\alpha$ and $F'|_{T_\alpha} = F_\alpha$. Then by Zorn's lemma there exists a maximal element in the whole set of pairs: $P_{max} = (T_{max}, F_{max})$. Let us show that

$T_{max} = \Psi$. If not, there exists $\gamma \in \Psi \setminus T_{max}$. Take an arbitrary $t \in T_{max}$ and consider the geodesic segment $[t, \gamma]$. The intersection $[t, \gamma] \cap T_{max}$ is either a closed segment $[t, t']$ or a semi-interval $[t, t')$. Consider each of these cases separately.

Case 1. $T_{max} \cap [t, \gamma] = [t, t')$. Take $T'_{max} = T_{max} \cup t'$. It is a good subtree since T'_{max} intersects just with a single connected component of $\Psi \setminus t'$ (since T_{max} is connected), and for the points from T_{max} the condition b) remains true. Due to completeness of A_μ, the isometric inclusion of T_{max} can be continued to the isometric inclusion of T'_{max}. This contradicts with the maximality of P_{max}.

Case 2. $T_{max} \cap [t, \gamma] = [t, t']$. Then T_{max} does not intersect with the connected component of $\Psi \setminus t'$ containing γ, therefore t' is few–valented. In each component of $\Psi \setminus t'$ non-intersecting with T_{max} we fix a point t_ν and consider geodesic segments $[t', t_\nu]$. Take $T'_{max} = \cup_\nu [t', t_\nu] \cup T_{max}$. This is a good subtree. Indeed, t' is full-valented. For the points $T_{max} \setminus t'$ the condition b) remains true, and the points of $T'_{max} \setminus T_{max} = \cup_\nu (t', t_\nu]$ are few–valented.

We build the continuation of F to T'_{max}. Consider $F(t') \in A_\mu$. The point $F(t')$ is an endpoint for μ rays lying in different connected components of $A_\mu \setminus F(t')$. Points of $F(T_{max})$ may lie in at most two of them. To every segment $[t', t_\nu]$ we isometrically correspond a segment $[F(t'), F(t_\nu)]$ on one of the "free" rays (i.e. having no points of T_{max}) of $A_\mu \setminus F(t')$. This is possible since cardinality of the set of the "free" rays is always greater or equal than of the set of points t_ν. Once again, we get a contradiction with the maximality of P_{max}.

Therefore, $T_{max} = \Psi$ and F_{max} is its isometric emebedding into the space A_μ. This completes the proof of (i).

(ii) Let Ψ be a complete real tree such that the valency of every its point has cardinality μ. We call the pair (T, F) *good*, if T is a good subtree and F preserves full-valentness. The same as in (i), the set of good pairs is non-empty and one may introduce partial order on it. It may be shown analogously that every linear ordered set of good pairs has a maximal element and hence by Zorn's lemma there exists a maximal pair (T_{max}, F_{max}), where $T_{max} = \Psi$. Let us prove that F_{max} is a surjection. If not, there exists $a \in A_\mu$ not lying in $F_{max}(T_{max})$. Take an arbitrary point $p \in F_{max}(T_{max})$ and consider $[a, p] \cap F_{max}(T_{max})$. Since $F_{max}(T_{max})$ is complete (being an isometric image of a complete metric space $T_{max} = \Psi$), it is a closed segment $[a, p']$. But in this case $F_{max}(T_{max})$ intersects not with all connected components of $A_\mu \setminus p'$ while $F_{max}^{-1}(p') \in \Psi$ is clearly full–valented — we got a contradiction with the fact that (T_{max}, F_{max}) is a good pair. Therefore, F_{max} is an isometry between A_μ and Ψ.

(iii) Let us note that in the proof of (ii) we could consider only such good pairs (T, F) that satisfy $(T_0, F_0) \leq (T, F)$, where $T_0 = \gamma_0$, $F_0(\gamma_0) = a_0$ for some arbitrary fixed $\gamma_0 \in \Psi$ and $a_0 \in A_\mu$. Then finally we would obtain an isometry between Ψ and A_μ which moves γ_0 into a_0. Setting $\Psi = A_\mu$ implies that A_μ is homogeneous. $\qquad\square$

2.3. **Asymptotic cones of manifolds of negative curvature.** In this section we prove Theorem 1.3.2 stating that any asymptotic cone of a manifold of negative curvature is isometric to the space A (see Definition 1.1.4).

Proof of Theorem 1.3.2. Let $O_i \in X$ be some sequence of basepoints and $\lambda_i \to \infty$ be some sequence of scaling factors. An asymptotic cone $\mathrm{Con}_\omega(X, O_i, \lambda_i)$ of a manifold of negative curvature X is a complete geodesic metric space. Let us show that the valency at every its point is $2^{\aleph_0}$.

Consider a point $\xi \in \mathrm{Con}_\omega(X, O_i, \lambda_i)$ and let $\{x_i\}$ be one of the sequences in X corresponding to this point. For every $\alpha \in [0, 2\pi)$ and for every $\rho > 0$ consider a sequence $\{x_i^{\alpha,\rho}\}$ constructed according to the following rule. Consider the geodesic segment $[O_i, x_i]$. The segment $[x_i, x_i^{\alpha,\rho}]$ is of length $\lambda_i \rho$ and the angle between it and $[O_i, x_i]$ is equal to α. It clearly follows from the triangle inequality that the distances between O_i and $x_i^{\alpha,\rho}$ grow not faster than linearly. Let us show that the distance between two points of the asymptotic cone ξ^{α_j, ρ_j} corresponding to the sequences $\{x_i^{\alpha_j, \rho_j}\}$, $j = 1, 2$, is equal to $\rho_1 + \rho_2$ if $\alpha_1 \neq \alpha_2$ and $|\rho_1 - \rho_2|$ if $\alpha_1 = \alpha_2$ The latter is due to the fact that $\alpha_1 = \alpha_2$ implies $d_X(x_i^{\alpha_1, \rho_1}, x_i^{\alpha_2, \rho_2}) = \lambda_i |\rho_1 - \rho_2|$. If $\alpha_1 \neq \alpha_2$ we need the following lemmas.

Lemma 2.3.1. *Let ABC and $A'B'C'$ be geodesic triangles in $\mathbb{H}$ and in X respectively, such that $|AB| = |A'B'| = u$, $|AC| = |A'C'| = v$, $|BC| = w$, $|B'C'| = w'$, $\angle A = \angle A' = \varphi$, where $|\cdot|$ denotes the length of a segment in the corresponding metric. Then $w < w'$.*

Proof. Consider a comparison triangle $PQR \in \mathbb{H}$ for ABC: $|PQ| = u$, $|PR| = v$, $|QR| = w$ and let $\angle P = \psi$. By Alexandrov comparison theorem (see [GhH] $\psi > \varphi$. On the other hand by hyperbolic cosine rule (see [Bea])

$$\cosh w = \cosh u \cosh v - \cos \varphi \sinh u \sinh v <$$

$$< \cosh u \cosh v - \cos \psi \sinh u \sinh v = \cosh w',$$

and hence $w < w'$. $\qquad\square$

Lemma 2.3.2. *Consider a point $O \in X$ and two geodesic rays l_1 and l_2 starting from O with a non-zero angle between them. Let $A_i \in l_1$*

and $B_i \in l_2$ be two sequences of points such that $d_X(O, A_i) = \lambda_i \rho_1$ and $d_X(O, B_i) = \lambda_i \rho_2$, where $\lambda_i \to \infty$. Then

$$\lim_{i \to \infty} \frac{d_X(A_i, B_i)}{\lambda_i} = \rho_1 + \rho_2. \qquad (2.3.3)$$

Proof. Let $O'A_i'B_i'$ be the triangle in $\mathbb{H}$ corresponding to OA_iB_i as in lemma 2.3.1. Then due to triangle inequality and lemma 2.3.1 we have:

$$\rho_1 + \rho_2 \geq \frac{d_X(A_i, B_i)}{\lambda_i} \geq \frac{d_{\mathbb{H}}(A_i', B_i')}{\lambda_i} \to \rho_1 + \rho_2,$$

where the limit at the right follows from a simple asymptotic analysis of distance formulas on the hyperbolic plane. This proves the relation (2.3.3). $\qquad \square$

Lemma 2.3.2 implies that

$$\frac{d_X(x_i^{\alpha_1, \rho_1}, x_i^{\alpha_2, \rho_2})}{\lambda_i} \to \rho_1 + \rho_2 \qquad (2.3.4)$$

Therefore $\{x_i^{\alpha_1, \rho_1}\} = \{x_i^{\alpha_2, \rho_2}\}$ iff $\alpha_1 = \alpha_2$ and $\rho_1 = \rho_2$. For every $\alpha \in [0, 2\pi)$ we get an infinite ray starting from the point ξ and for different α these rays have the only point ξ in common and hence lie in different connected components of $\mathrm{Con}_\omega(X, O_i, \lambda_i) \setminus \xi$, i.e. the valency of the point ξ is $2^{\aleph_0}$. Since ξ is an arbitrary point in the asymptotic cone $\mathrm{Con}_\omega(X, O_i, \lambda_i)$ we have proved that it is isometric to the metric space A. $\qquad \square$

3. Proof of Theorem 1.2.3

3.1. **Part(b).** We start with the easier part of Theorem 1.2.3, namely, we prove that the space A can be isometrically embedded at infinity into a non-abelian free group.

Let us fix some sequence of scaling factors $\lambda_i \to \infty$ (for simplicity we set $\lambda_i = i$).

Consider a Cayley graph corresponding to a subgroup generated by two arbitrary generators γ_1, γ_2 of the group Γ, and let G be its part corresponding to non-negative powers of γ_1, γ_2. Consider the standard coding of vertices of G with binary sequences of γ_1-s and γ_2-s taking the unity as the initial vertex. Setting $\gamma_1 = 0$ and $\gamma_2 = 1$ we correspond to every vertex of G a binary number in a unique way.

For any point $t \in [0, \rho_f)$ let $[t_l, t_r) \ni t$ be the maximal interval where the function $f \in A$ is constant. For every $\varepsilon > 0$ consider the following "cut-off" function: $f^\varepsilon(t) = f(t)$ if $t_r - t_l \geq \varepsilon$ and $f^\varepsilon(t) = 0$ otherwise.

Fix some bijection F from $[0, 1)$ to the set of all binary sequences. Our aim is to build a sequence of vertices $\{x_i\} \in G$ (which are actually

elements of the group Γ) corresponding to an arbitrary function $f \in A$. Consider a function $f^{1/\sqrt{i}}$, and let $l_1^i, ..., l_{k_i}^i$ be the lengths of the segments on which it is constant, and $0 = a_1^i, ..., a_{k_i}^i$ be the values of $f^{1/\sqrt{i}}$ on them. Consider the binary sequence $F(a_1^i)$ and take its first $[l_1 \cdot i]$ elements, where $[\cdot]$ denotes the integer part. We get a binary number which corresponds to a unique point $x_1^i \in G$. From the point x_1^i we go along the sequence $F(a_2^i)$ for the time $[l_2 \cdot i]$ and so on until $x_{k_i}^i = x_i$. Let us check that

$$\frac{d_G(x_i, e)}{i} \to \rho_f.$$

Note that $k_i \leq 2\sqrt{i}\rho_f + 1$ since the number of intervals on which the function $f^{1/\sqrt{i}}$ does not vanish is not greater than $\sqrt{i} \cdot \rho_f$. Therefore

$$d_G(x_i, e) = \sum_{r=1}^{k_i}[l_r^i \cdot i] \geq \sum_{r=1}^{k_i}(l_r^i \cdot i - 1) = \rho_f \cdot i - k_i.$$

Thus,

$$\rho_f = \sum_{r=1}^{k_i} l_r^i \geq \frac{d_G(x_i, e)}{i} \geq \rho_f - \frac{k_i}{i},$$

and since $k_i/i \to 0$ we get $d_G(x_i, e)/i \to \rho_f$.

Consider now two arbitrary functions $f, g \in A$. Let s be their separation moment, and let s_i be moments of separation of the segments $[e, x_i^f]$ and $[e, x_i^g]$. Note that for sufficiently small ε the moment of separation between f^ε and g^ε coincides with the separation moment of f and g, and $f^\varepsilon|_{[0,s]} = (f|_{[0,s]})^\varepsilon$. Applying previous arguments to the function $f|_{[0,s]}$ we get that $d_G(s_i, e)/i \to s$, since for the sufficiently large i we have $s_i = x_i^{f|_{[0,s]}}$. Hence

$$\frac{d_G(x_i^f, x_i^g)}{i} = \frac{d_G(x_i^f, e)}{i} + \frac{d_G(x_i^g, e)}{i} - \frac{d_G(s_i, e)}{i} \to \rho_f + \rho_g - 2s = d_A(f, g).$$

This completes the proof of part (b) of the theorem. $\qquad\square$

3.2. **Part(a).** In this section we prove that the metric space A can be isometrically embedded at infinity into any manifold of negative curvature. First let us mention that in case of the hyperbolic plane $\mathbb{H}$ (or, more general, in case of a complete simply connected manifold of *constant* negative curvature) one may isometrically embed the space A into it in a very explicit and simple way. Consider the Poincare disc model of $\mathbb{H}$ and let us introduce the hyperbolic polar coordinates (r, φ) centered at some $O \in \mathbb{H}$. To every function $f(t) \neq f_\emptyset \in A$,

$f : [0, \rho) \to [0, 1)$ we correspond a sequence $x_n = (\rho_i, \varphi_i) \in \mathbb{H}$, where $\rho_i = \rho i$ and

$$\varphi_i = 2\pi f(0) + \int_0^\rho e^{-ti} f(t) dt.$$

To the "empty function" $f_\emptyset$ we correspond a constant sequence $x_i = O$. We also set $\lambda_i = i$. Using some asymptotic analysis of distance formulas on the hyperbolic plane one can prove that this is indeed an isometric embedding at infinity.

Let us prove the theorem in the general case. We divide the proof into several steps.

Step 1. Let us fix a sequence of scaling factors (as before, let $\lambda_i = i$). Let $O \in X$ be a basepoint and $l_0 \in X$ some fixed geodesic ray emanating from O. For every $0 \le \varphi < 2\pi$ let us fix a ray $l_\varphi \subset X$ starting from the point O such that the angle between $\angle(l_0, l_\varphi) = \varphi$. As in the case of the hyperbolic plane, let us correspond to $f_\emptyset$ the constant sequence $x_i = O$.

We prove that for any $f : [0, \rho) \to [0, 1)$ and for any $i > 0$ there exists a unique naturally–parametrized path $\gamma_i : [0, \rho \cdot i] \to X$ with the following properties:

(i) if a, b are such that $f|_{[a,b]} = $ const. then $\gamma_i|_{[a \cdot i, b \cdot i]}$ is a geodesic segment;

(ii) if a is a point of discontinuity of the function f then the angle between $[O, \gamma_i(a \cdot i)]$ and $[\gamma_i(a \cdot i), \gamma_i((a + \varepsilon)i)]$ is equal to $1/100 + f(a)$, where ε is such that $f|_{[a,a+\varepsilon]} = $ const;

(iii) if $f|_{[0,a]} \equiv f(0)$ then $\gamma(a) \in l_{2\pi f(0)}$.

Step 2. There is always a unique way to start constructing γ due to the property (iii), since for every function there exists a non–zero interval $[0, \varepsilon]$ where it is constant and is equal to $f(0)$. If γ is constructed on $[0, r \cdot i)$ it may be continued in a unique way to the point r by setting $\gamma_i(ri) = \lim_{z \to ri-0} \gamma_i(z)$. If γ_i is constructed on $[0, r \cdot i]$ for some $r < \rho$ we may take such $\varepsilon > 0$ that $f|_{[r,r+\varepsilon]} = $ const. There are two possible cases: if r is a point of discontinuity of f then we apply (ii) in order to continue γ_i up to the point $(r + \varepsilon)i$, and if f is continuous at r we may apply (i). Therefore it is possible to construct a unique path γ_i on the whole $[0, \rho i]$. This path is a broken geodesic line with possibly infinite number of links.

We set $\{x_i^f\} = \gamma_i(\rho \cdot i)$. Let us prove that the correspondence $f \in A \to \{x_i^f\} \subset X$ satisfies Definition 1.2.1.

Step 3. Our next aim is to show that for any $f, g \in A$ we have

$$\frac{d_X(x_i^f, x_i^g)}{i} \to d_A(f, g).$$

Let us prove first that

$$\frac{d_X(x_i^f, O)}{i} \to \rho. \tag{3.2.1}$$

Denote a function $f_r = f|_{[0,r]}$. Consider the set

$$R = \{r \in [0, \rho)| \,\forall r' \le r \quad \frac{d_X(x_i^{f_r}, O)}{i} \to r'\}.$$

We want to verify that $R = [0, \rho]$. We do it in the same way as we have shown the existence of the path γ_i in the previous Step.

Since for some $\varepsilon > 0$ $f|_{[0,\varepsilon]} = $ const. therefore $\varepsilon \in R$. If every $r_1 < r$ belongs to R then $r \in R$. This follows from the triangle inequality:

$$r \ge \frac{d_X(x_i^{f_r}, O)}{i} \ge \frac{d_X(x_i^{f_{r_1}}, O)}{i} - (r - r_1),$$

and since we may choose r_1 arbitrary close to r we get

$$\frac{d_X(x_i^{f_r}, O)}{i} \to r.$$

Finally, if $r \in R$ for some $r < \rho$ then $r + \varepsilon \in R$ for some $\varepsilon > 0$. Indeed, let r' be the left end of the interval containing r on which f is constant. Due to (ii) the angle $\angle([O, \gamma_i(r')], [\gamma_i(r'), \gamma_i(r+\varepsilon)]) = 1/100 + f(r') > 0$ and hence using Lemma 2.3.2 we obtain that $d_X(x_i^{f_{r+\varepsilon}}, O)/i \to r + \varepsilon$. This completes the proof of the fact that $R = [0, \rho]$.

Step 4. In order to proceed we need the following lemma.

Lemma 3.2.2. *Let $A_i B_i C_i$ be a triangle in X such that $|A_i B_i| = u \cdot i$, $|A_i C_i| = v \cdot i$, $u, v > 0$. Consider a path ζ_i of length $|\zeta_i|$ connecting C_i and B_i such that*

$$\lim_{i \to \infty} \frac{u \cdot i}{|\zeta_i| + v \cdot i} = 1. \tag{3.2.3}$$

Then $\lim_{i \to \infty} \angle A_i = 0$.

Proof. Suppose that the angle $\angle A_i$ does not tend to zero. Then there exist a subsequence $i_k \to \infty$ such that $\angle A_{i_k} > \alpha > 0$. Then due to Lemma 2.3.2 $d_X(B_{i_k}, C_{i_k})/i_k \to u + v$. On the other hand, $|\zeta_{i_k}| \ge d_X(B_{i_k}, C_{i_k})$ and hence

$$\lim_{i_k \to \infty} \frac{|\zeta_{i_k}| + v \cdot i_k}{u \cdot i_k} \ge \frac{u + 2v}{u} > 1,$$

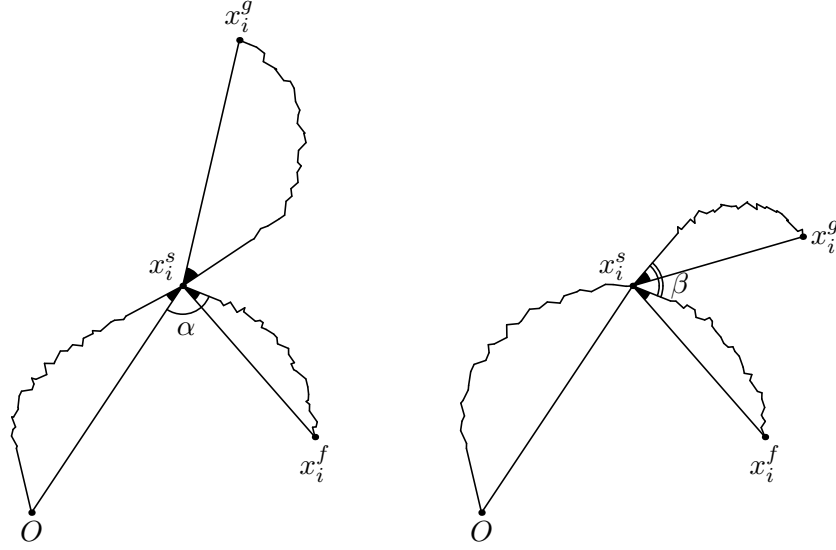which contradicts with (3.2.3). This completes the proof of Lemma 3.2.2. $\square$

FIGURE 1.

Step 5. Consider a function $f \in A$ which is discontinious at the point s. We say that it has a *discontinuity of the first type* at s if there exists $\delta > 0$ such that $f_{[s-\delta,s]} \equiv$ const. Otherwise we say that f has a *discontinuity of the second type* at s. Actually, discontinuities of the second type are accumulation points of discontinuities of the first type.

Now, let $f, g \in A$ be arbitrary functions and s be their separation moment. At least one of the functions should be discontinuous at s, let it be the function f. Note that if it has a discontinuity of the first type at s, then g is either continuous, or has a discontinuity of the first type, and if f has a discontinuity of the second type at s, the function g has also a discontinuity of the second type at this point.

One may check in each case that due to the property (ii) and Lemma 3.2.2 the paths corresponding to functions f and g for large i separate exactly at the point $x_i^s = x_i^{f_s} = x_i^{g_s}$, and that the angle $\angle([x_i^s, x_i^f], [x_i^s, x_i^g])$ is separated from zero. Indeed, if g is continuous at s then f has a discontinuity of the first type (see Figure 1, left); the angles marked with black are close to zero by Lemma 3.2.2, the angle α by property (ii) is neither 0 nor π and does not depend on i.

If g is discontinuous at s (see Figure 1, right) then similarly the angles marked with black are close to zero by Lemma 3.2.2 and the angle β by property (ii) is neither 0 nor π and does not depend on i. Thus it can be shown analogously to (3.2.1) that $d_X(O, x_i^s)/i \rightarrow s$, $d_X(x_i^s, x_i^f)/i \rightarrow \rho_f - s$, $d_X(x_i^s, x_i^g)/i \rightarrow \rho_g - s$. Therefore, applying

Lemma 2.3.2 we finally obtain:

$$\lim_{i\to\infty}\frac{d_X(x_i^f, x_i^g)}{i} = \lim_{i\to\infty}\frac{d_X(x_i^f, x_i^s) + d_X(x_i^s, x_i^g)}{i} = (\rho_f - s) + (\rho_g - s) = d_A(f, g),$$

which completes the proof of the theorem.   □

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